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93202Q

932022

NEW ZEALAND QUALIFICATIONS AUTHORITY
MANA TOHU MĀTAURANGA O AOTEAROA

Scholarship 2009
Mathematics with Calculus

9.30 am Saturday 21 November 2009
Time allowed: Three hours
Total marks: 40

QUESTION BOOKLET

Pull out the Formulae and Tables Booklet S–CALCF from the centre of this booklet.

There are FIVE questions in this booklet. Answer ALL questions.

Write your answers in the Answer Booklet 93202A.

Show all working. Start your answer to each question on a new page. Carefully number each question.

Answers developed using a CAS calculator require **ALL commands to be shown**. Correct answers only will not be sufficient.

Check that this booklet has pages 2–6 in the correct order and that none of these pages is blank.

YOU MAY KEEP THIS BOOKLET AT THE END OF THE EXAMINATION.

You have three hours to complete this examination.

QUESTION ONE (8 marks)

(a) Suppose a, b and k are constants.

Show that $y = a\sin(kx) + b\cos(kx)$ is a solution to the differential equation $\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2} = -k^2 y$.

(b) The position of a point on the edge of a circle of radius 1 rolling along the x-axis is given by the parametric equations $x = t + \sin t$ and $y = 1 + \cos t$.

The curve is called a **cycloid**, and is shown in Figure 1 below for $-4\pi \leq t \leq 4\pi$.

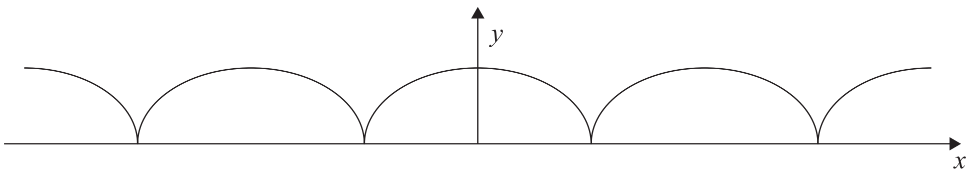

Figure 1: The cycloid formed by rolling a circle of radius 1.

The speed of the point is given by $s(t) = \sqrt{\left(\dfrac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\dfrac{\mathrm{d}y}{\mathrm{d}t}\right)^2}$

and the scalar acceleration of the point is given by $a(t) = \dfrac{\mathrm{d}s}{\mathrm{d}t}$.

Find and simplify an expression for $s(t)$.

Find an expression for $a(t)$, and hence show that $a(t)$ is not continuous.

(c) Consider the complex equation $(1 + \operatorname{cis} 2\theta)(1 + \operatorname{cis} 4\theta) = u + \mathrm{i}v$.

Show that $\dfrac{u}{v} = \cot 3\theta$.

QUESTION TWO (8 marks)

(a) The **hyperbolic cosine** function is defined as $\cosh x = \dfrac{e^x + e^{-x}}{2}$.

Prove that $\cosh 3x = 4(\cosh x)^3 - 3\cosh x$.

(b) Suppose $p > 0$ and $a > 0$. The curve $y = x^p$ between $(0,0)$ and (a,a^p) is rotated once around the x-axis and once around the y-axis to make two separate volumes of revolution.

What values of a (in terms of p) make the two separate solids of equal volume?

(c) Show that Simpson's Rule, using $n = 2$, gives an exact answer when integrating any quadratic $y = ax^2 + bx + c$ on the interval $r \le x \le t$.

QUESTION THREE (8 marks)

(a) An error sometimes made by students learning about complex numbers is to write $\dfrac{1}{a+\mathrm{i}b}=\dfrac{1}{a}+\dfrac{1}{\mathrm{i}b}$,

where a and b are real numbers, and $a+\mathrm{i}b \neq 0$.

Show that $\dfrac{1}{a+\mathrm{i}b} \neq \dfrac{1}{a}+\dfrac{1}{\mathrm{i}b}$ for the above conditions.

(b) Six points are shown in the Argand diagram in Figure 2. They are the roots of $p(x)$, a degree 6 polynomial with real coefficients.

The points lie on two concentric circles centred at the origin, and are the vertices of equilateral triangles, as shown in the figure.

The positive real root of $p(x)$ is k.

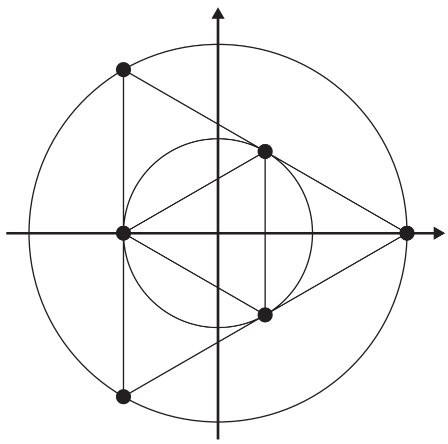

Figure 2: Argand diagram showing the roots of $p(x)$.

(i) List the **exact** roots of $p(x) = 0$. Hence or otherwise write $p(x)$.
It need not be expanded, but should not contain complex terms.

(ii) Find the **exact** roots of $p'(x)$ and indicate their approximate position on a sketch of Figure 2.

QUESTION FOUR (8 marks)

(a) The **arc length** of a differentiable function y between $x = a$ and $x = b$ is given by:

$$L = \int_a^b \sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x$$

The equation $y = k - \sqrt{x^3}$ forms a curve between the y-axis and the x-axis, where k, x and y are positive.

Find the value of k that makes the arc length of this curve exactly $\frac{56}{27}$.

Give k in **surd form**.

(b) A sector, of angle θ, is cut from a circle of radius R, as shown in Figure 3.

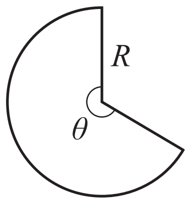

Figure 3: A sector cut from a circle of radius R.

Find the value of θ that maximises the volume of the cone which the sector is formed into.
You need not prove that your answer gives a maximum.

(c) A conical perfume bottle tapers to a point at the top as shown in Figure 4.

The bottle has been left open, and the rate of evaporation of the perfume is proportional to the exposed surface area of the remaining perfume.

The bottle begins $\frac{7}{8}$ full, and the perfume takes 10 days to evaporate completely.

When is the bottle half-full?

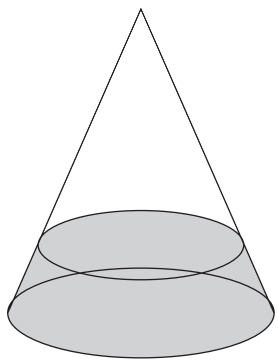

Figure 4: A partly-full conical perfume bottle.

QUESTION FIVE (8 marks)

(a) An ellipse is drawn around a square so that the minor axis of the ellipse is a diagonal of the square, and the area of the ellipse is twice the area of the square, as shown in Figure 5.

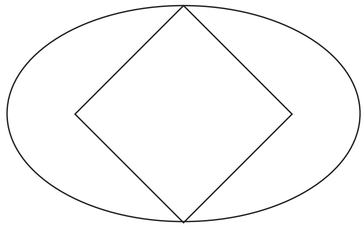

Figure 5: Square inside ellipse.

The area of the ellipse with equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $A=\pi ab$.

Find the eccentricity of the ellipse.

(b) Consider the **hyperbola** $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

Given that the equation for the tangent at the point (x_0, y_0) is $\frac{xx_0}{a^2}-\frac{yy_0}{b^2}=1$,

show that the tangent to the hyperbola at (x_0, y_0) does not intersect the curve anywhere else.

(c) Consider the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

The point Q lies at the intersection of two perpendicular tangents to the ellipse.

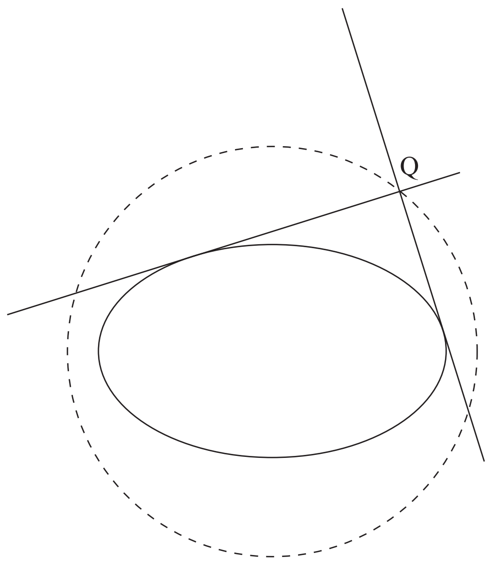

Figure 6: The circular locus of the point Q.

Given that the equation for a tangent to the ellipse, with gradient m, is $y = mx \pm \sqrt{a^2m^2+b^2}$, show that the locus of Q is a circle (see Figure 6).